Solving cubic equations

1 Introduction

Recall that **quadratic** equations can easily be solved, by using the quadratic formula. In particular, we have

$$ax^2 + bx + c = 0 \text{ if and only if } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

The expression $b^2 - 4ac$ is known as the **discriminant** of the quadratic, and is sometimes denoted by Δ. We have the following three cases:

Case I: If $\Delta > 0$, the quadratic equation has two real solutions.

Case II: If $\Delta = 0$, the quadratic equation has only one real solution.

Case III: If $\Delta < 0$, the quadratic equation has no real solutions.

The corresponding formulae for solving cubic and quartic equations are significantly more complicated, (and for polynomials of degree 5 or more, there is no general formula at all)!!

In the next section, we shall consider the formulae for solving cubic equations. Later, in Section 3, we shall also consider a numerical method for giving **approximate** solutions to a wide range of equations (including cubic equations).

2 The cubic formula

In this section, we investigate how to find the **real** solutions of the cubic equation

$$x^3 + ax^2 + bx + c = 0.$$

Step 1.

First we let

$$p = b - \frac{a^2}{3} \quad \text{and} \quad q = \frac{2a^3}{27} - \frac{ab}{3} + c$$

Then we define the **discriminant** Δ of the cubic as follows:

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}$$

Step 2.

We have the following three cases:

Case I: $\Delta > 0$. In this case there is **only one real solution.** It is given by

$$x = \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{\frac{1}{3}} - \frac{a}{3}$$

Case II: $\Delta = 0$. In this case there are repeated roots. The roots are given by

$$x_1 = -2\left(\frac{q}{2}\right)^{\frac{1}{3}} - \frac{a}{3} \qquad \text{and} \qquad x_2 = x_3 = \left(\frac{q}{2}\right)^{\frac{1}{3}} - \frac{a}{3}$$

Case III: $\Delta < 0$. In this case there are **three** real solutions:

$$\begin{aligned}
x_1 &= \frac{2}{\sqrt{3}}\sqrt{-p}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{3\sqrt{3}q}{2(\sqrt{-p})^3}\right)\right) - \frac{a}{3} \\
x_2 &= -\frac{2}{\sqrt{3}}\sqrt{-p}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{3\sqrt{3}q}{2(\sqrt{-p})^3}\right) + \frac{\pi}{3}\right) - \frac{a}{3} \\
x_3 &= \frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\sin^{-1}\left(\frac{3\sqrt{3}q}{2(\sqrt{-p})^3}\right) + \frac{\pi}{6}\right) - \frac{a}{3}
\end{aligned}$$

Example 1. Find all real solutions to

$$x^3 - \sqrt{3}x^2 - 2x + 2\sqrt{3} = 0.$$

Solution:

We have $a = -\sqrt{3}$, $b = -2$ and $c = 2\sqrt{3}$.

Thus

$$p = b - \frac{a^2}{3} = -3 \quad \text{and} \quad q = \frac{2a^3}{27} - \frac{ab}{3} + c = \frac{10\sqrt{3}}{9}$$

and so

$$\begin{aligned} \Delta &= \frac{q^2}{4} + \frac{p^3}{27} \\ &= \frac{(\frac{10\sqrt{3}}{9})^2}{4} + \frac{(-3)^3}{27} \\ &= -\frac{2}{27} \\ &< 0. \end{aligned}$$

We therefore have three real solutions. Since

$$\sin^{-1}\left(\frac{3\sqrt{3}q}{2(\sqrt{-p})^3}\right) = \sin^{-1}\left(\frac{5}{3\sqrt{3}}\right),$$

we have

$$\begin{aligned} x_1 &= \frac{2}{\sqrt{3}}\sqrt{3}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{5}{3\sqrt{3}}\right)\right) - \frac{-\sqrt{3}}{3} \\ &= 2\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{5}{3\sqrt{3}}\right)\right) + \frac{\sqrt{3}}{3} = 1.414213562 \qquad \text{(9 d.p.)} \\ x_2 &= -\frac{2}{\sqrt{3}}\sqrt{3}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{5}{3\sqrt{3}}\right) + \frac{\pi}{3}\right) - \frac{-\sqrt{3}}{3} \\ &= -2\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{5}{3\sqrt{3}}\right) + \frac{\pi}{3}\right) + \frac{\sqrt{3}}{3} = -1.414213562 \qquad \text{(9 d.p.)} \\ x_3 &= \frac{2}{\sqrt{3}}\sqrt{3}\cos\left(\frac{1}{3}\sin^{-1}\left(\frac{5}{3\sqrt{3}}\right) + \frac{\pi}{6}\right) - \frac{-\sqrt{3}}{3} \\ &= 2\cos\left(\frac{1}{3}\sin^{-1}\left(\frac{5}{3\sqrt{3}}\right) + \frac{\pi}{6}\right) + \frac{\sqrt{3}}{3} = 1.732050808 \qquad \text{(9 d.p.)} \end{aligned}$$

Example 2. Find all real solutions to

$$x^3 - 4x^2 + 5x - 2 = 0.$$

Solution:

We have $a = -4, \;\; b = 5$ and $c = -2$.

Thus

$$p = b - \frac{a^2}{3} = -\frac{1}{3} \qquad \text{and} \qquad q = \frac{2a^3}{27} - \frac{ab}{3} + c = -\frac{2}{27}$$

and so

$$\begin{aligned} \Delta &= \frac{q^2}{4} + \frac{p^3}{27} \\ &= \frac{(-\frac{2}{27})^2}{4} + \frac{(-\frac{1}{3})^3}{27} \\ &= 0. \end{aligned}$$

Hence there are repeated roots.

The roots are given by

$$\begin{aligned} x_1 &= -2\left(\frac{q}{2}\right)^{\frac{1}{3}} - \frac{a}{3} \\ &= -2\left(-\frac{1}{27}\right)^{\frac{1}{3}} - \frac{-4}{3} \\ &= -2 \times -\frac{1}{3} + \frac{4}{3} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{and } x_2 = x_3 &= \left(\frac{q}{2}\right)^{\frac{1}{3}} - \frac{a}{3} \\ &= \left(-\frac{1}{27}\right)^{\frac{1}{3}} - \frac{-4}{3} \\ &= -\frac{1}{3} + \frac{4}{3} \\ &= 1 \end{aligned}$$

Example 3. Find all real solutions to

$$x^3 + x - 2 = 0.$$

Solution:

We have $a = 0, \;\; b = 1$ and $c = -2$.

Thus

$$p = b - \frac{a^2}{3} = 1 \text{ and } q = \frac{2a^3}{27} - \frac{ab}{3} + c = -2$$

and so

$$\begin{aligned} \Delta &= \frac{q^2}{4} + \frac{p^3}{27} \\ &= \frac{(-2)^2}{4} + \frac{(1)^3}{27} \\ &= 1 + \frac{1}{27} \\ &= \frac{28}{27} \\ &> 0. \end{aligned}$$

Therefore, we have only one real solution. It is given by

$$\begin{aligned} x &= \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{\frac{1}{3}} - \frac{a}{3} \\ &= \left(-\frac{-2}{2} + \sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} + \left(-\frac{-2}{2} - \sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} - \frac{0}{3} \\ &= \left(1 + \sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} + \left(1 - \sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \\ &= 1 \qquad \text{(We will prove this in Section 4.)} \end{aligned}$$

2.1 Exercises

Find all real solutions to the following equations:

(a) $x^3 - 6.5x^2 + 12.87x - 7.623 = 0$

(b) $x^3 - \dfrac{x^2}{3} - \dfrac{x}{4} + \dfrac{1}{12} = 0$

(c) $x^3 - \sqrt{2}x^2 - 3x + 3\sqrt{2} = 0$

(d) $x^3 + \dfrac{x^2}{2} + \dfrac{x}{2} - \dfrac{1}{2} = 0$

(e) $x^3 + (1 - \sqrt{3})x^2 + (2 - \sqrt{3})x - 2\sqrt{3} = 0$

(f) $x^3 + 0.7x^2 + 2.7x - 0.9 = 0$

(g) $x^3 - \dfrac{7}{\sqrt{3}}x^2 + 5x - \sqrt{3} = 0$

(h) $x^3 - \dfrac{7}{6}x^2 + \dfrac{4}{9}x - \dfrac{1}{18} = 0$

(i) $x^3 - 0.4x^2 + 0.05x - 0.002 = 0$

(j) $x^3 + x^2 + x + 2 = 0$

(k) $2x^3 + 10x^2 - 2x - 4 = 0$

(l) $x^3 - 9x + 4 = 0$

Answers:

(a) $1.1,\ 2.1,\ 3.3$

(b) $0.5,\ -0.5,\ \frac{1}{3}$

(c) $\sqrt{2},\ \sqrt{3},\ -\sqrt{3}$

(d) 0.5

(e) $\sqrt{3}$

(f) 0.3

(g) $\frac{1}{\sqrt{3}},\ \sqrt{3}$

(h) $\frac{1}{3},\ \frac{1}{2}$

(i) $0.1,\ 0.2$

(j) -1.35321

(k) $-0.568372,\ -5.11902,\ 0.687399$

(l) $0.454903,\ -3.20147,\ 2.74656$

3 Newton's Method

Note, this section requires knowledge of *derivatives!* If you have not learnt any calculus before, then you might want to postpone this section until Term 2.

- Read the section on Newton's Method in the textbook by Stewart.
- Use Newton's Method to solve each of the cubic equations given in Section 2.1 above.

4 Proof of Earlier Result

On the bottom of page 5, we wrote that

$$\left(1+\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} + \left(1-\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \;=\; 1$$

We will now prove that result.

In our proof, we will be using the following two facts:

Result 1:

$$\left(1+\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \;=\; \frac{1}{2}+\frac{\sqrt{7}}{2\sqrt{3}}$$

Proof of Result 1:

Since

$$\begin{aligned}
\left(\frac{1}{2}+\frac{\sqrt{7}}{2\sqrt{3}}\right)^3 \;&=\; \left(\frac{1}{2}\right)^3 \;+\; 3\times\left(\frac{1}{2}\right)^2\times\frac{\sqrt{7}}{2\sqrt{3}} \;+\; 3\times\frac{1}{2}\times\left(\frac{\sqrt{7}}{2\sqrt{3}}\right)^2 \;+\; \left(\frac{\sqrt{7}}{2\sqrt{3}}\right)^3 \\
&=\; \frac{1}{8} \;+\; \frac{3\sqrt{7}}{8\sqrt{3}} \;+\; \frac{3}{2}\times\frac{7}{4\times 3} \;+\; \frac{7\sqrt{7}}{8\times 3\sqrt{3}} \\
&=\; \frac{1}{8} \;+\; \frac{3\sqrt{7}}{8\sqrt{3}} \;+\; \frac{7}{8} \;+\; \frac{7\sqrt{7}}{24\sqrt{3}} \\
&=\; \frac{8}{8} \;+\; \frac{16\sqrt{7}}{24\sqrt{3}} \\
&=\; 1 \;+\; \frac{2\sqrt{7}}{3\sqrt{3}} \\
&=\; 1 \;+\; \frac{\sqrt{28}}{\sqrt{27}}
\end{aligned}$$

then

$$\frac{1}{2}+\frac{\sqrt{7}}{2\sqrt{3}} \;=\; \left(1+\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}}$$

as claimed.

Similarly, we have:

Result 2:

$$\left(1-\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} = \frac{1}{2}-\frac{\sqrt{7}}{2\sqrt{3}}$$

Proof of Result 2:

Since

$$\begin{aligned}\left(\frac{1}{2}-\frac{\sqrt{7}}{2\sqrt{3}}\right)^3 &= \left(\frac{1}{2}\right)^3 - 3\times\left(\frac{1}{2}\right)^2\times\frac{\sqrt{7}}{2\sqrt{3}} + 3\times\frac{1}{2}\times\left(\frac{\sqrt{7}}{2\sqrt{3}}\right)^2 - \left(\frac{\sqrt{7}}{2\sqrt{3}}\right)^3 \\
&= \frac{1}{8} - \frac{3\sqrt{7}}{8\sqrt{3}} + \frac{3}{2}\times\frac{7}{4\times 3} - \frac{7\sqrt{7}}{8\times 3\sqrt{3}} \\
&= \frac{1}{8} - \frac{3\sqrt{7}}{8\sqrt{3}} + \frac{7}{8} - \frac{7\sqrt{7}}{24\sqrt{3}} \\
&= \frac{8}{8} - \frac{16\sqrt{7}}{24\sqrt{3}} \\
&= 1 - \frac{2\sqrt{7}}{3\sqrt{3}} \\
&= 1 - \frac{\sqrt{28}}{\sqrt{27}}\end{aligned}$$

then

$$\frac{1}{2}-\frac{\sqrt{7}}{2\sqrt{3}} = \left(1-\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}}$$

as claimed.

We can now easily prove our main result, as shown below:

Main Result:

$$\left(1+\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} + \left(1-\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \;=\; 1$$

Proof of the Main Result:

$$\begin{aligned}\left(1+\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} + \left(1-\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \;&=\; \frac{1}{2}+\frac{\sqrt{7}}{2\sqrt{3}} \quad + \quad \frac{1}{2}-\frac{\sqrt{7}}{2\sqrt{3}} \quad \text{(by Results 1 and 2)} \\ &=\; 1, \quad \text{as required.}\end{aligned}$$

5 Where did this proof come from?

In this section, we see how Results 1 and 2 were "discovered".

Suppose that

$$\left(1+\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \;=\; m \qquad \text{and} \qquad \left(1-\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \;=\; n.$$

Then

$$1+\sqrt{\frac{28}{27}} \;=\; m^3 \tag{1}$$

and

$$1-\sqrt{\frac{28}{27}} \;=\; n^3. \tag{2}$$

Furthermore,

$$m > 0 \qquad \text{and} \qquad n < 0.$$

In our main result, we wanted to prove that

$$\left(1+\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} + \left(1-\sqrt{\frac{28}{27}}\right)^{\frac{1}{3}} \;=\; 1.$$

Thus we wanted m and n to satisfy

$$m+n \;=\; 1.$$

That is, we wanted to have

$$n = 1 - m. \tag{3}$$

From Equations 1 and 2 we have

$$\begin{aligned} m^3 + n^3 &= 2 \\ \text{Then (using Equation 3) we have} \quad m^3 + (1-m)^3 &= 2. \\ \text{That is, we have} \quad m^3 + 1 - 3m + 3m^2 - m^3 &= 2. \\ \text{That is } \; 3m^2 - 3m - 1 &= 0. \end{aligned}$$

By the Quadratic Formula we obtain

$$\begin{aligned} m &= \frac{3 \;\pm\; \sqrt{21}}{6} \\ &= \frac{3}{6} \;\pm\; \frac{\sqrt{7 \times 3}}{2 \times 3} \\ &= \frac{1}{2} \;\pm\; \frac{\sqrt{7}}{2\sqrt{3}} \end{aligned}$$

Since $m > 0$, we choose

$$m = \frac{1}{2} \;+\; \frac{\sqrt{7}}{2\sqrt{3}}\;.$$

Then Equation 3 gives us

$$\begin{aligned} n &= 1 - \left(\frac{1}{2} + \frac{\sqrt{7}}{2\sqrt{3}}\right) \\ &= \frac{1}{2} \;-\; \frac{\sqrt{7}}{2\sqrt{3}} \end{aligned}$$

Note:

> In Results 1 and 2 it was checked that the values of m and n (as found above) are correct!