Making Pi

1 Introduction

1.1 Definition of Pi

Consider any circle with radius $r$ and circumference $C$. In the following sections, we show that the ratio

$$\frac{C}{2r}$$

is just a number which does not change with different circles. This is a very useful fact, since once we know what the number is, we can use it to find $C$ whenever we know the value $r$. We define $\pi$ to be this special ratio:

$$\pi = \frac{C}{2r}.$$

Then we automatically get the formula $C = 2\pi r$, which enables us to evaluate $C$ whenever we know the value of $r$.

1.2 Evaluation of Pi

It can be shown that the number $\pi$ is irrational (in fact transcendental); this means that the decimal expansion of $\pi$ goes on forever and does not repeat itself, and so we can never evaluate $\pi$ exactly! What we do here is we find a sequence of better and better approximations for $\pi$. To do this, we use the same method used by Archimedes (about 200 B.C.).
2 Archimedes’ Method for finding $\pi$

We first find approximations for the circumference $C$ of a circle. We can approximate the value of $C$ with the perimeter of a polygon inscribed inside the circle. In the following diagram, we have a hexagon inscribed inside a circle of radius $r$. The hexagon is made up of six equilateral triangles, each of which has side length $r$. Now

$$C \approx r \times 6 = 6r,$$
and so $\pi \approx \frac{6r}{2r} = 3$.

To improve this approximation for $\pi$, we continually double the number of sides of the polygon inscribed inside the circle.

Then we get an increasing sequence

$$C_0 = \ell_0 \times 6, \quad C_1 = \ell_1 \times 6 \times 2, \quad C_2 = \ell_2 \times 6 \times 2^2, \ldots, \quad C_n = \ell_n \times 6 \times 2^n, \ldots$$

of approximations for $C$. In fact, we actually define the circumference $C$ by $C = \lim_{n \to \infty} C_n$. Archimedes did not have a computer, calculator or even tables of the trigonometric functions (sine, cosine and tangent) at his disposal to calculate $\ell_n$; so how did he calculate $\ell_n$ without using trigonometry?
We start with $\ell_0 = r$ (in the six-sided polygon). In order to find $\ell_1$ we will use a $30^\circ$ right-angled triangle. This can be found as follows.

In the following diagram, the angle $C\overline{P_0D}$ is a right-angle (by the Circle Lemma (a)). Also, by the Circle Lemma (b), the angle $\angle CD\overline{P_0} = \frac{1}{2} 60^\circ = 30^\circ$. Thus, we can construct a $30^\circ$ right-angled triangle, with side lengths $r$, $\frac{r}{2}$ and $\frac{\sqrt{3}}{2}r$, as shown in the following diagram:

We use the $30^\circ$ right-angled triangle to find $\ell_1$:

Note that we can calculate $\ell_1$ by using Pythagoras’ Theorem:

$$\ell_1^2 = \left(\frac{r}{2}\right)^2 + \left(r - \frac{\sqrt{3}}{2}r\right)^2 = \frac{r^2}{4} + r^2 - \sqrt{3}r^2 + \frac{3}{4}r^2 = (2 - \sqrt{3})r^2.$$  

Therefore, without using trigonometry, we have calculated that $\ell_1 = r\sqrt{2 - \sqrt{3}}$. 

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As a bonus, we get even more information: we can now construct a $15^\circ$ right-angled triangle. In the following diagram, the angle $CP_1D$ is a right-angle (by the Circle Lemma (a)). Also, by the Circle Lemma (b), the angle $CDP_1 = \frac{1}{2}30^\circ = 15^\circ$. Thus, we can construct a $15^\circ$ right-angled triangle, with side lengths $r$, $\frac{\ell_1}{2}$ and $\frac{1}{2}\sqrt{4r^2 - \ell_1^2}$, as shown in the following diagram:

We can use this $15^\circ$ right-angled triangle to calculate $\ell_2$.

Now

$$
\ell_2^2 = \left(\frac{\ell_1}{2}\right)^2 + \left(r - \frac{1}{2}\sqrt{4r^2 - \ell_1^2}\right)^2
$$

$$
= \frac{\ell_1^2}{4} + r^2 - r\sqrt{4r^2 - \ell_1^2} + \frac{1}{4}(4r^2 - \ell_1^2)
$$

$$
= 2r^2 - r^2 \sqrt{4 - \left(\frac{\ell_1}{r}\right)^2}
$$

$$
= r^2 \left(2 - \sqrt{4 - \left(\frac{\ell_1}{r}\right)^2}\right)
$$
and so \( \ell_2 = r\sqrt{2 - 4 - \left(\frac{l_1}{r}\right)^2} \).

If we repeat this process by constructing a 7.5° right-angled triangle, we get

\[
\ell_3^2 = \left(\frac{\ell_2}{2}\right)^2 + \left(r - \frac{1}{2}\sqrt{4r^2 - \ell_2^2}\right)^2 = 2r^2 - r\sqrt{4r^2 - \ell_2^2} = r^2 \left(2 - \sqrt{4 - \left(\frac{\ell_2}{r}\right)^2}\right)
\]

and so \( \ell_3 = r\sqrt{2 - 4 - \left(\frac{l_2}{r}\right)^2} \).

In general, for any natural number \( i \), we have

\[
\ell_{i+1} = r\sqrt{2 - 4 - \left(\frac{l_i}{r}\right)^2}.
\]

We now have a method for calculating \( \ell_i \):

\[
\begin{align*}
\ell_0 &= r \\
\ell_1 &= r\sqrt{2 - 4 - \left(\frac{l_0}{r}\right)^2} = r\sqrt{2 - \sqrt{4 - 1^2}} = r\sqrt{2 - \sqrt{3}} \\
\ell_2 &= r\sqrt{2 - 4 - \left(\frac{l_1}{r}\right)^2} = r\sqrt{2 - \sqrt{2 + \sqrt{3}}} \\
\ell_3 &= r\sqrt{2 - 4 - \left(\frac{l_2}{r}\right)^2} = r\sqrt{2 - \sqrt{2 + 2 + \sqrt{3}}} \\
\ell_4 &= r\sqrt{2 - 4 - \left(\frac{l_3}{r}\right)^2} = r\sqrt{2 - \sqrt{2 + 2 + 2 + \sqrt{3}}}.
\end{align*}
\]

Now, since \( C_n = \ell_n \times 6 \times 2^n \) (see page 2), we have the following approximations for \( C \):

\[
\begin{align*}
C_0 &= \ell_0 \times 6 = 6r \\
C_1 &= \ell_1 \times 6 \times 2 = 12r\sqrt{2 - \sqrt{3}} \\
C_2 &= \ell_2 \times 6 \times 2^2 = 24r\sqrt{2 - \sqrt{2 + \sqrt{3}}} \\
C_3 &= \ell_3 \times 6 \times 2^3 = 48r\sqrt{2 - \sqrt{2 + 2 + \sqrt{3}}} \\
C_4 &= \ell_4 \times 6 \times 2^4 = 96r\sqrt{2 - \sqrt{2 + 2 + 2 + \sqrt{3}}}.
\end{align*}
\]

Recall that, on page 2, we defined \( C \) by \( C = \lim_{n \to \infty} C_n \). This means that

\[
\pi = \frac{C}{2r} = \frac{1}{2r} \lim_{n \to \infty} C_n = \lim_{n \to \infty} \frac{C_n}{2r}.
\]
Therefore, if we let $\pi_n = \frac{C_n}{2r}$, then the sequence $\pi_0, \pi_1, \pi_2, \ldots$ approaches $\pi$ as $n \to \infty$. Our approximations for $\pi$ are now

$$
\begin{align*}
\pi_0 &= \frac{C_0}{2r} = \frac{6r}{2r} = 3 \\
\pi_1 &= \frac{C_1}{2r} = \frac{12r\sqrt{2 - \sqrt{3}}}{2r} = 6\sqrt{2 - \sqrt{3}} = 3.105829 \\
\pi_2 &= \frac{C_2}{2r} = 12\sqrt{2 - \sqrt{2 + \sqrt{3}}} = 3.132629 \\
\pi_3 &= \frac{C_3}{2r} = 24\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} = 3.139350 \\
\pi_4 &= \frac{C_4}{2r} = 48\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} = 3.141032.
\end{align*}
$$

Since each $\pi_i$ ($i = 0, 1, 2, 3, \ldots$) is just a real number (independent of $r$ and $C$), we have that $\pi$ is also just a real number, as stated in Section 1.

### 2.1 Archimedes’ Calculation

When Archimedes found approximations for $\pi$, he first needed to find approximations for the square roots of numbers (because he didn’t have a calculator!). At the time, various methods for finding square roots were known (see the Maths 1 Extension topic “How does a calculator calculate $\sin x$?”). He actually found the approximation $3 + \frac{10}{71}$ for $\pi_4$. Furthermore, he found approximations for the circumference $C$ of a circle by circumscribing the circle with polygons, i.e., by putting the polygons on the outside of the circle. These other approximations are always greater than $C$, whereas the approximations that we have found are always smaller than $C$. By comparing the two different approximations, we can get an upper bound and a lower bound for $\pi$. When $n = 4$, Archimedes found that

$$
3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}.
$$

It is interesting to note that $3 + \frac{1}{7} = \frac{22}{7}$ is a very common approximation for $\pi$ that is still used today!

Compare the following values:

$$
\begin{align*}
3 + \frac{10}{71} &= 3.140845 \\
\pi_4 &= 3.141032 \\
\pi &= 3.14159265358979 \\
\frac{22}{7} &= 3.142857.
\end{align*}
$$

### 2.2 Rounding Errors

Note that $C_4$ is the perimeter of a polygon with 96 sides. If we calculated $C_5$, it would be the perimeter of a polygon with $96 \times 2 = 192$ sides, and so $C_5$ is a better approximation that $C_4$. 

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In general $C_{n+1}$ is a better approximation that $C_n$, and so $\pi_{n+1}$ is a better approximation that $\pi_n$. In theory, this means that we can compute $\pi$ to any degree of accuracy we like, by choosing $n$ large enough. Unfortunately, this does not work in practice.

Consider the formula $C_n = \ell_n \times 6 \times 2^n$. When we calculate $\ell_n$ with a calculator or computer, we can only do so for a fixed number of decimal places; this causes rounding error. For example, suppose that we calculate $\ell_{20}$ with an error of 0.00000000001. Since $C_{20} = \ell_{20} \times 6 \times 2^{20}$, we will have an error of 0.000000000001 $\times$ 6 $\times$ 2$^{20}$ = 0.000006291456 for $C_{20}$.

As $n$ gets even larger the term $2^n$ causes small errors in $\ell_n$ to become large errors in $C_n$. This means that Archimedes’ Method is not good for approximating $\pi$ to a large number of decimal places with a computer or calculator.

For example, using about 14 decimal places we have

$$\begin{align*}
\pi_0 &= 3 \\
\pi_5 &\approx 3.14145247228546 \\
\pi_{10} &\approx 3.14145247228546 \\
\pi_{15} &\approx 3.1415926534921 \\
\pi_{20} &\approx 3.14159264532122 \\
\pi_{25} &\approx 3.14167426502176 \\
\pi_{30} &\approx 3.18198051533946 \\
\pi_{31} &\approx 3 \\
\pi_{32} &\approx 4.24264068711929.
\end{align*}$$

The true value of $\pi$ is $\pi = 3.14159265358979$, and so our approximations $\pi_i$ start to get worse when $i \approx 20$.

### 3 Modern methods for finding $\pi$

For centuries, people have been fascinated by $\pi$ and a lot of effort has been put into computing $\pi$ to as many decimal places as possible (its decimal expansion never repeats).

To 39 decimal places of accuracy, it can be shown that

$$\pi = 3.141592653589793238462643383279502884197$$

But how can we find $\pi$ so accurately? One of the most common methods for computing $\pi$ is called Machin’s Formula.

#### 3.1 Machin’s Formula

One method of calculating $\pi$ is to use the facts that

\begin{enumerate}
  \item $\frac{\pi}{4} = \tan^{-1}(1)$; and
  \item inverse tangent can be expanded in a power series:
  \[\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x| \leq 1.\]
\end{enumerate}
It follows that \( \pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) \). Unfortunately, the above series converges quite slowly. For example
\[
4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}\right) = 3.28373848373848.
\]
With this series, we need a lot of terms to evaluate \( \pi \) accurately.
To get a series that converges faster, we note that the series
\[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\]
converges faster when \( x \) is closer to zero. Machin realised that
\[
\tan^{-1}(1) = 4 \tan^{-1} \left(\frac{1}{5}\right) - \tan^{-1} \left(\frac{1}{239}\right)
\]
(see Section 4 for details) and that the series for \( \tan^{-1} \left(\frac{1}{5}\right) \) and \( \tan^{-1} \left(\frac{1}{239}\right) \) converge quite fast. Thus Machin’s Formula:
\[
\frac{\pi}{4} = 4 \tan^{-1} \left(\frac{1}{5}\right) - \tan^{-1} \left(\frac{1}{239}\right)
\]
can be used to approximate \( \pi \) fairly accurately.

**Example 1.** We have
\[
\tan^{-1} \left(\frac{1}{5}\right) \approx \frac{1}{5} - \frac{(\frac{1}{5})^3}{3} + \frac{(\frac{1}{5})^5}{5} = 0.197397333333333
\]
and
\[
\tan^{-1} \left(\frac{1}{239}\right) \approx \frac{1}{239} - \frac{(\frac{1}{239})^3}{3} + \frac{(\frac{1}{239})^5}{5} = 0.00418407600207473.
\]
Therefore an approximate value for \( \pi \) is given by
\[
\pi = 16 \tan^{-1} \left(\frac{1}{5}\right) - 4 \tan^{-1} \left(\frac{1}{239}\right)
\approx 16(0.197397333333333) - 4(0.00418407600207473)
= 3.14162102932503.
\]

### 3.2 Other formulas

There are hundreds of formulas for calculating \( \pi \). Some interesting ones are

\[
(i) \quad \frac{\pi}{2} = \prod_{n=1}^{\infty} \left[\frac{(2n)^2}{(2n-1)(2n+1)}\right] = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}
\]

\[
(ii) \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots
\]

\[
(iii) \quad \frac{\pi}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(n!)^2 2^{n+1}}{(2n+1)!} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots
\]
4 Exercises

Machin-like formulas come from the identity
\[
\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}.
\]

Let \( x = \tan \theta_1 \) and \( y = \tan \theta_2 \). Then
\[
\tan(\theta_1 + \theta_2) = \frac{x + y}{1 - xy}
\]
and so
\[
\theta_1 + \theta_2 = \tan^{-1} \left( \frac{x + y}{1 - xy} \right).
\]

Since \( x = \tan \theta_1 \) and \( y = \tan \theta_2 \), we have \( \theta_1 = \tan^{-1}(x) \) and \( \theta_2 = \tan^{-1}(y) \).

Thus
\[
\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left( \frac{x + y}{1 - xy} \right). \quad (1)
\]

We can use the above identity to find numbers \( x \) and \( y \) such that \( \tan^{-1} x + \tan^{-1} y = \tan^{-1}(1) \).

Our goal is to make \( x \) and \( y \) as close to zero as possible, so that the power series for \( \tan^{-1} x \) converges quickly.

(a) **(Euler’s Formula.)** Use identity (1) and the fact that \( \frac{\pi}{4} = \tan^{-1}(1) \) to show that
\[
\tan^{-1} \left( \frac{1}{2} \right) + \tan^{-1} \left( \frac{1}{3} \right) = \frac{\pi}{4}.
\]

(b) **(Hutton’s Formula.)** We now find \( x \) such that \( \tan^{-1} \left( \frac{1}{2} \right) = \tan^{-1} x + \tan^{-1} \left( \frac{1}{3} \right) \).

By identity (1), the number \( x \) must satisfy
\[
\tan^{-1} \left( \frac{1}{2} \right) = \tan^{-1} \left( \frac{x + \frac{1}{3}}{1 - \frac{x}{3}} \right).
\]

Solve the equation
\[
\frac{1}{2} = \frac{x + \frac{1}{3}}{1 - \frac{x}{3}}
\]
for \( x \), and then by using (a), show that
\[
\frac{\pi}{4} = 2 \tan^{-1} \left( \frac{1}{3} \right) + \tan^{-1} \left( \frac{1}{7} \right).
\]

(c) Show that
\[
\alpha = \frac{x + y}{1 - xy} \quad \Rightarrow \quad x = \frac{\alpha - y}{1 + \alpha y}. \quad (2)
\]
(d) By using (1) and (2) find $x$ such that $\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1} x + \tan^{-1} \left(\frac{1}{7}\right)$; hence by using (b) show that
\[ \frac{\pi}{4} = 3\tan^{-1} \left(\frac{1}{7}\right) + 2\tan^{-1} \left(\frac{4}{22}\right) . \]

Identity (1) implies two other identities:
\[ \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x - y}{1 + xy}\right) \]  
\[ 2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2}\right) \]  

These identities can be used to derive more Machin-like formulas:

(e) **(Hermann’s Formula.)** Use (3) and (4) to show that
\[ \frac{\pi}{4} = 2\tan^{-1} \left(\frac{1}{2}\right) - \tan^{-1} \left(\frac{1}{7}\right) . \]

(f) **(Machin’s Formula.)** Use (4) to show that $4\tan^{-1}(\frac{4}{5}) = \tan^{-1} \left(\frac{120}{119}\right)$. Now use (3) to show that
\[ \tan^{-1} \left(\frac{120}{119}\right) - \tan^{-1} \left(\frac{1}{239}\right) = \tan^{-1}(1) . \]

Hence deduce that
\[ \frac{\pi}{4} = 4\tan^{-1} \left(\frac{1}{5}\right) - \tan^{-1} \left(\frac{1}{239}\right) . \]

5 **Reference**

A large amount of information about $\pi$ can be viewed on the internet at:

- http://mathworld.wolfram.com/Pi.html
  (Don’t type www before mathworld!)